PRIMENESS IN NEAR-RINGS WITH
MULTIPLICATIVE SEMI-GROUP SATISFYING
‘THE THREE IDENTITIES’

A. O. ATAGÜN and N. J. GROENEWALD

Department of Mathematics
Bozok University
66100, Yozgat
Turkey
e-mail: aosman.atagun@bozok.edu.tr

Department of Mathematics
University of Port Elizabeth
6000
South Africa

Abstract
In this paper, the interconnections of completely prime, 3-prime and equiprime
developments are considered in right permutable, left permutable and medial near-rings.
Some results for right self distributivity, left self distributivity and insertion
factors in near-rings are given.

1. Introduction

Throughout this paper, all near-rings are right near-rings. This paper
considers primeness in near-rings with the multiplicative semi-group
satisfying one of the following identities:

a. \( abc = acb \) (right permutable near-rings)
b. $abc = bac$ (left permutable near-rings)

c. $abcd = acbd$ (medial near-rings).

Birkenmeier and Heatherly [6] called these “the three identities”. They developed a theory of rings satisfying the three identities [5]. A perusal of the near-ring literature reveals many types of near-rings which satisfy one of the three identities. Pilz [12] used the phrase “weakly commutative” for “right permutability” in near-rings. Near-rings which are both right permutable and left permutable are called permutable. Also playing a role in this paper are the identities:

d. $abc = acbc$ (right self-distributive (RSD))

e. $abc = abac$ (left self-distributive (LSD)).


An ideal $I$ of a near-ring $N$ is called a completely prime ideal of $N$ if whenever $ab \in I$, then $a \in I$ or $b \in I$. The study of completely prime ideals in near-rings goes back at least to [13], where such an ideal is called a “prime ideal of type 2”. The ideal $I$ is said to be completely semi-prime if $a^2 \in I$ implies $a \in I$. In [13] Ramakotaiah and Rao defined the concept of a prime ideal of type 1. An ideal $I$ of $N \langle I \triangleleft N \rangle$ is prime of type 1 if for all $x, y \in N \times N \subseteq I$ implies $x \in I$ or $y \in I$. Groenewald [8] used the phrase “3-prime ideal” for “prime ideal of type 1”. An ideal $I$ is a 3-semiprime ideal if whenever $x N x \subseteq I$, then $x \in I$. Booth et al. [7] gave another generalization of prime rings which they called equiprimeness. $P \triangleleft N$ is called equiprime if $a, x, y \in N \ anx - any \in P$ for all $n \in N \ implies \ a \in P \ or \ x - y \in P$. If $P$ is equiprime, then it is 3-prime. If the zero ideal of $N$ is 3-prime (resp. completely prime, equiprime), then we say $N$ is a 3-prime (resp., completely prime, equiprime) near-ring.

Birkenmeier and Heatherly [4] showed that 3-prime (3-semiprime) ideals in an LSD or RSD near-ring are also completely prime.
In [6], these authors proved that 3-prime ideals in a medial near-ring are also completely prime.

The main aim of this paper is to find the conditions which satisfy 3-primeness implies equiprimeness in near-rings with the identities $a, b, c, d \text{ and } e$.

For all undefined terms in near-rings, the reader may refer to Pilz [12].


In general, completely primeness doesn’t imply equiprimeness. For example, if $(N, +)$ is any cyclic group of prime order $p (p > 2)$, define $ab = a$ if $b \neq 0$ and $ab = 0$ if $b = 0$, then $N$ is a near-ring which is completely prime but not equiprime [7].

For a near-ring $N$, the distributive part of $N$ is the set $\{d \in N : d$ is distributive$\}$ and denoted by $N_d$.

**Lemma 2.1.** Let $N$ be near-ring and $P \triangleleft N$. Then $N_d P \subseteq P$.

**Proof.** Let $n_d \in N_d$. Then $n_d 0 = n_d (0 + 0) = n_d 0 + n_d 0$, i.e., $n_d 0 = 0$. Since $P \triangleleft N$, $n_d p = n_d (p + 0) - n_d 0 \in P$ for every $p \in P$.

**Proposition 2.2.** Let $N$ be a right permutable near-ring and $P \triangleleft N$. Then $P$ is 3-prime if and only if $P$ is completely prime.

**Proof.** Note that for any near-ring a completely prime ideal is a 3-prime ideal. Assume $N$ is right permutable, $xy \in P$ and $P$ is a 3-prime ideal. Then $xyN^2 = xNyN \subseteq P$. Now either $x \in P$ or $yN \subseteq P$. If $yN \subseteq P$, then $yN \subseteq P$. Hence $y \in P$. Thus $P$ is completely prime.

**Theorem 2.3.** Let $N$ be a right permutable near-ring and let $P \triangleleft N$ be such that $N_d \setminus P \neq 0$. If $P$ is a completely prime ideal, then $P$ is equiprime.
Proof. Suppose that for \( a, x, y \in N, anx - any \in P \) for every \( n \in N \). Since \( N \) is right permutable, then \( anx - any = axn - ayn = (ax - ay)n \in P \). Then \( ax - ay \in P \) or \( n \in P \), since \( P \) is completely prime. If \( n \in P \), then \( N = P \), which is a contradiction of choice of \( P \) because \( N_d \setminus P \neq \emptyset \) and \( N_d \setminus P \subseteq N \). Hence \( ax - ay \in P \). By Lemma 2.1. for an \( n \in N_d \setminus P, n_d(ax - ay) = n_dax - n_day \in P \). Since \( N \) is right permutable, \( n_dax - n_day = n_dxa - n_dya = n_d(x - y)a \in P \). Then \( n_d(x - y) \in P \) or \( a \in P \), since \( P \) is completely prime. Since \( n_d \notin P \), \( x - y \in P \) or \( a \in P \). Therefore \( P \) is an equiprime ideal of \( N \).

**Corollary 2.4.** Let \( N \) be a right permutable near-ring and let \( P \triangleleft N \) be such that \( N_d \setminus P \neq \emptyset \). Then \( P \) is 3-prime if and only if \( P \) is equiprime.

**Proof.** Since every equiprime ideal is also 3-prime, the proof is seen from Proposition 2.2 and Theorem 2.3.

Theorem 2.3. is illustrated by the following example:

**Example** (cf. [1]). Let the additive group \((\mathbb{Z}_6, +)\). Under a multiplication given in the following table, \((\mathbb{Z}_6, +, \cdot)\) is a right permutable near-ring. We say \( N = (\mathbb{Z}_6, +, \cdot) \).

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Let \( P = \{0, 3\} \). Then \( P \) is an ideal of \( N \) and \( 4 \in N_d \setminus P \), i.e., \( N_d \setminus P \neq \emptyset \). Then \( N \) and \( P \) satisfy the conditions of Theorem 2.3. It is seen that \( P \) is completely prime and equiprime.
Proposition 2.5. Let $N$ be a left permutable near-ring and $P \triangleleft N$ be such that $NP \subseteq P(\text{or } P^2 = P)$. Then $P$ is 3-prime if and only if $P$ is completely prime.

Proof. Assume $N$ is left permutable, $xy \in P$ and $P$ is a 3-prime ideal such that $NP \subseteq P$. Then $Nxy = xNy \subseteq P$. Thus $P$ is completely prime (If $P^2 = P$, then for every $n \in N$ and for every $p \in P$, $np = np_1p_2 = p_1np_2 \in P$, i.e., $NP \subseteq P$).

Theorem 2.6. Let $N$ be a left permutable near-ring and let $P$ be a proper ideal of $N$. If $P$ is completely prime, then it is equiprime.

Proof. For $a, x, y \in N$, assume $anx - any \in P$ for all $n \in N$. Since $N$ is left permutable, then $nax - nay \in P$. Let $q \in N - P$. Then $(nax - nay)q = naxq - nayq = nxaq - nyaq = (nx - ny)aq \in P$. Since $N$ is completely prime and $q \notin P$, then $(nx - ny)a \in P$. Now either $nx - ny \in P$ or $a \in P$. If $nx - ny \in P$, then $(nx - ny)q = nxq - nyl = xnl - ynl = (x - y)nl \in P$. Then $(x - y)Nl \subseteq P$. Since $P$ is also 3-prime and $q \notin P$, then $x - y \in P$. Thus $P$ is equiprime.

Corollary 2.7. Let $N$ be a left permutable near-ring and let $P$ be a proper ideal of $N$ such that $NP \subseteq P(\text{or } P^2 = P)$. Then $P$ is 3-prime if and only if $P$ is equiprime.

Proof. It is seen from Proposition 2.5 and Theorem 2.6.

Proposition 2.8. Let $N$ be a permutable near-ring and $P \triangleleft N$. Then $P$ is 3-prime if and only if $P$ is equiprime.

Proof. For $a, x, y \in N$, assume $anx - any \in P$ for all $n \in N$. Since $N$ is permutable, then $anx - any = axn - ayn = xan - yan = xna - yna = (x - y)na \in P$. So $(x - y)Na \subseteq P$. Then either $x - y \in P$ or $a \in P$. Thus $P$ is equiprime, the converse is straightforward.
Proposition 2.9 ([6, Proposition 2.7]). Let \( N \) be a medial near-ring and \( P \triangleleft N \). Then \( P \) is 3-prime if and only if \( P \) is completely prime.

Theorem 2.10. Let \( N \) be a medial near-ring and \( P \triangleleft N \) be such that \( N_d \setminus P \neq \emptyset \). If \( P \) is completely prime, then it is equiprime.

Proof. For \( a \in N - P \), \( x, y \in N \), assume \( anx - any \in P \) for all \( n \in N \). Let \( n_d \in N_d - P \). Since \( N \) is medial and \( N_d P \subseteq P \) by Lemma 2.1, then \( n_d anx - n_d any = n_d anx - n_d any \in P \). Then, \( n_d anx an_d - n_d any an_d = (n_d anx - n_d any) an_d \in P \). Since \( P \) is completely prime and \( an_d \notin P \), then \( n_d anx - n_d any \notin P \). Then \( n_d anx an_d - n_d any an_d = n_d anx n_d - n_d any n_d = n_d(x - y) n_d \in P \). Then \( n_d(x - y) N_n d \subseteq P \). Since \( P \) is completely prime and \( n_d \notin P \), then \( x - y \notin P \). Therefore \( P \) is equiprime.

Corollary 2.11. Let \( N \) be a medial near-ring and \( P \triangleleft N \) be such that \( N_d \setminus P \neq \emptyset \). Then \( P \) is 3-prime if and only if \( P \) is equiprime.

Proof. The proof is seen from Proposition 2.9 and Theorem 2.10.

\( \beta_i(N) \) will denote the intersections of all \( i \)-prime ideals of \( N \) for \( i = 3, \text{equi}, c \) = completely. For more details on the prime radicals \( \beta_i(N) \), the following papers: [2] and [9] are recommended. We have the following:

Corollary 2.12. Let \( N \) be a zero-symmetric near-ring. If \( \beta_c(N) \) is a 0-prime ideal of \( N \), then

(a) If \( N \) is a right permutable (or medial) near-ring and there exists a completely prime ideal \( P \) of \( N \) such that \( N_d \setminus P \neq \emptyset \), then \( \beta_e(N) \subseteq \beta_c(N) \).

(b) If \( N \) is a left permutable near-ring and \( N \neq P \) is a completely prime ideal of \( N \), then \( \beta_e(N) \subseteq \beta_c(N) = \beta_3(N) \).

(c) If \( N \) is a permutable near-ring, then \( \beta_e(N) = \beta_c(N) = \beta_3(N) \).
Proof. (a) It is easily seen that $\beta_c(N)$ is a completely semi-prime ideal of $N$. Under assumptions $\beta_c(N)$ is a 0-prime ideal of $N$, then $\beta_c(N)$ is a completely prime ideal of $N$ from [9]. Since $P$ is a completely prime ideal of $N$ such that $N_d \setminus P \neq \emptyset$ and $\beta_c(N) \subseteq P$, then $N_d \setminus \beta_c(N) \neq \emptyset$. Hence $\beta_c(N)$ is an equiprime ideal of $N$ from Theorem 2.3 (for medial, from Theorem 2.10). Therefore $\beta_e(N) \subseteq \beta_c(N). \beta_c(N) = \beta_3(N)$ comes from Proposition 2.2 (for medial, from Proposition 2.9).

(b) $\beta_e(N)$ is a completely prime ideal of $N$ by the proof of (a). Since $N \neq P$ is a completely prime ideal of $N$ and $\beta_c(N) \subseteq P$, then $\beta_c(N) \neq N$. Hence $\beta_e(N)$ is an equiprime ideal of $N$ from Theorem 2.6, i.e., $\beta_e(N) \subseteq \beta_c(N)$. Since $N$ is a zero-symmetric near-ring, then $N\beta_e(N) \subseteq \beta_e(N)$. Therefore $\beta_c(N) = \beta_3(N)$ by Proposition 2.5.

(c) $\beta_c(N)$ is a completely prime ideal of $N$ by the proof of (a). Hence the result follows from Proposition 2.8.

3. Insertion Factors, LSD and RSD Near-Rings

In a near-ring $N$, an element $x \in N$ is called an insertion factor in $N$ if for every $a, b \in N$ with $a b = axb$. Throughout this section, $I$ denotes the set of all insertion factors in $N$. Every constant near-ring is an example of $I = N$.

If $n \in N$, the annihilator of $n$ is $(0 : n) = \{x \in N : x n = 0\}$.

Lemma 3.1. Let $N$ be a near-ring.

(i) If $(0 : n) = 0$ for all $n$ in an RSD near-ring $N$, then $I = N$.

(ii) If $I = N$, then $N$ is both RSD and LSD.

(iii) If $N$ is RSD, then $N$ has strong IFP property [4, Lemma 2.8].

(iv) If $N$ is RSD and simple, then $I = N$.

(v) If $N$ is RSD and simple, then $N$ is LSD.
Let $I = N$ and $P \triangleleft N$. Then $P$ is 3-prime if and only if $P$ is completely prime.

**Proof.** (i) Assume $N$ is RSD and for every $n \in N(0 : n) = 0$. Then for every $a, b, x \in N$, $abx = axbx$. Then $ab - axb \in (0 : x) = 0$. Hence $I = N$.

(ii) Suppose that $I = N$. Then for every $a, b, c \in N$ $ab = acb$. Hence $abc = acbc$, i.e., $N$ is RSD. Similarly $N$ is LSD.

(iv) If $N$ is RSD and simple, then either $0 : n = 0$ or $0 : n = N$ for all $n \in N$ as a result of (iii). If for all $n \in N(0 : n) = 0$, $I = N$ by (i). If for all $n \in N(0 : n) = N$, then $ab = 0 = anb$ for all $a, b, n \in N$. Hence $I = N$.

(v) From (iv) and (ii).

(vi) Assume $I = N$, $xy \in P$ and $P$ is a 3-prime ideal. Then $xy = xny \in P$ for every $n \in N$. So $xNy \subseteq P$. Thus $P$ is completely prime. The converse is straightforward.

**Theorem 3.2.** Let $I = N$ and $P$ an ideal of $N$ such that $N_d \setminus P \neq 0$. Then $P$ is 3-prime if and only if $P$ is equiprime.

**Proof.** Assume $P$ is a completely prime ideal of $N$ and for $a \in N - P$, $x, y \in N$ $anx - any \in P$ for every $n \in N$. Since $N_dP \subseteq P$ by Lemma 2.1., then for $n_d \in N_d - P$ $n_danx - n_dany \in P$. Since $P \triangleleft N$, $n_danxn_d - n_danyn_d = n_dxanxn_d - n_dyanyn_d \in P$. By Lemma 3.1 (ii), $N$ is LSD, then $x(an)n_d = x(an)n_d$ and $y(an)n_d = y(an)n_d$. Then $n_danxn_d - n_dany$ $n_d = n_dxanxn_d - n_dyanyn_d = n_dxann_d - n_dannn_d = n_d(x - y)ann_d \in P$. So $n_d(x - y)anN_d \subseteq P$. Since $P$ is completely prime and $a, n_d \notin P$, then $x - y \in P$. Thus $P$ is equiprime. The remain of the proof is seen from Lemma 3.1 (vi).
References


